Thermal Eqbrm → i) D in FDT → how do collective modes damp?.

Waves and Landau Damping in Collisionless Plasma.

→ Phase space flow incompressible (Liouville Thm.)

→ Derive Vlasov Eqn. from:

Words on the origin of Vlasov Eqn.

- Liouville Eqn.
- $\underset{\text{"peas"}}{N} = \sum_i \delta(\underline{x}-\underline{x}_i)\,\delta(\underline{v}-\underline{v}_i)$

Klimontovich Eqn.

- hierarchy, with $f(\underline{X}_1, \underline{X}_2, t) =$

"crushed pea soup" ← $f(\underline{X}_1, t)\, f(\underline{X}_2, t) + g(\underline{X}_1, \underline{X}_2, t)$

and $1/n\lambda_D^3 \ll 1 \Rightarrow g \ll f^2$ etc.

(Return in Fluctuations Discussion)

IV.) Collective Response in Collisionless Plasma

→ Waves in Vlasov Plasma (1D)

- $\omega, kv \gg \nu \Rightarrow$

$$f = \langle f \rangle + \tilde{f}$$

Physics and Results of Landau Pblm.

$$\langle f \rangle = (1/\sqrt{2\pi}\, v_{Th})\exp(-v^2/2v_{Th}^2)$$

(Maxwellian)

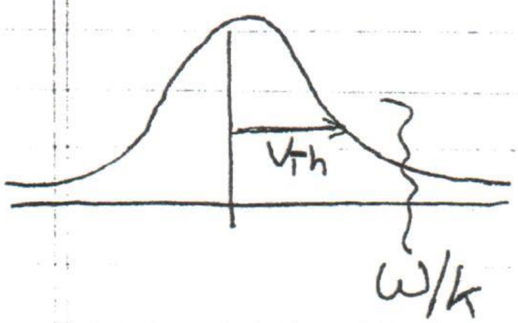
i.e. $\langle f \rangle$ established on long-time scale

i.e. Liouville Eqn → BBGKY ($1/n\lambda_D^3 \ll 1$) → Boltzmann Eqn. → Vlasov Eqn.

- Seek contact with Langmuir Wave (ions stationary)

$\Rightarrow \omega > k v_{th}$

(Heuristic)



Then, linearize:

$$\frac{\partial \tilde{f}}{\partial t} + v \frac{\partial \tilde{f}}{\partial x} = -\frac{q}{m} \tilde{E} \frac{\partial \langle f \rangle}{\partial v}$$

$$\nabla^2 \tilde{\phi} = -4\pi n_0 q \int \tilde{f} \, dv$$

$$f = \sum_{k,\omega} f_{k,\omega} \, e^{i(kx - \omega t)}$$

$$\Rightarrow \; -i(\omega - kv) \tilde{f}_{k,\omega} = \frac{q}{m} ik \phi_{k,\omega} \, \partial \langle f \rangle / \partial v$$

$$+ \; k^2 \tilde{\phi}_{k,\omega} = 4\pi n_0 q \int \tilde{f}_{k,\omega} \, dv$$

$$\tilde{f}_{k,\omega} = -k \frac{q}{m} \frac{\tilde{\phi}_{k,\omega} \, \partial \langle f \rangle / \partial v}{(\omega - kv)}$$

so
$$k^2 \tilde{\phi}_{k,\omega} = -\omega_p^2 k \int dv \, \frac{\partial \langle f \rangle / \partial v}{(\omega - kv)} \, \tilde{\phi}_{k,\omega}$$

Thus, $\epsilon(k,\omega) = 1 + \frac{\omega_p^2}{k}\int dv \frac{\partial\langle f\rangle/\partial v}{(\omega - kv)}$

– dielectric function for Vlasov Plasma

$\boxed{\text{? How Handle Pole at } \omega = kv \text{ ?}}$

– Recall V. E. derived in limit $\gamma \to 0$

$1/\omega - kv = \lim_{\epsilon\to 0} 1/\omega - kv + i\epsilon$

(Landau)

Concepts
– wave-particle resonance
– collisionless damping

– Alternatively, causality requires: $\tilde{\phi} \underset{t\to -\infty}{\to} 0$

$\phi \sim e^{-i\omega t} \Rightarrow \phi \sim e^{-i(\omega + i\epsilon)t}$

$\therefore$

(i.e. formally IVP)

$1/\omega - kv = \lim_{\epsilon\to 0} 1/\omega - kv + i\epsilon$

$= \frac{P}{\omega - kv} - i\pi\,\delta(\omega - kv)$

(Plemelj Formulae)

Fields Medal → Villani, Nonlinear Problem.

$$\epsilon(k,\omega) = 1 + \frac{\omega_p^2}{k}\int dv \, \frac{\partial\langle f\rangle/\partial v}{\omega - kv}$$

$$= 1 + \frac{\omega_p^2}{k}\int dv \, \frac{P}{\omega - kv}\frac{\partial\langle f\rangle}{\partial v}$$

$$- i\pi \frac{\omega_p^2}{k|k|}\frac{\partial\langle f\rangle}{\partial v}\Big|_{\omega/k}$$ → physical content !?

i.e. $$\delta(\omega - kv) = \frac{1}{|k|}\delta(v - \omega/k)$$

Further: $$\frac{\partial\langle f\rangle}{\partial v} = -\frac{v}{v_{Th}}\langle f\rangle$$

$$kv_{Th} < \omega \Rightarrow \frac{P}{\omega - kv} = \frac{1}{\omega}\left(1 + \frac{kv}{\omega} + \left(\frac{kv}{\omega}\right)^2 + \left(\frac{kv}{\omega}\right)^3 + \dots\right)$$

$$\epsilon_r(k,\omega) = 1 - \frac{\omega_p^2}{kv_{Th}^2}\int \frac{\langle f\rangle v}{\omega}\left(1 + \frac{k}{\omega}v + \left(\frac{kv}{\omega}\right)^2 + \left(\frac{kv}{\omega}\right)^3 + \dots\right)$$

$$= 1 - \frac{\omega_p^2}{\omega^2} - 3\omega_p^2\frac{v_{Th}^2 k^2}{\omega^4}$$

N.B. $\langle x^4 \rangle = \int dx \, x^4 \, e^{-x^2/2}$

$$= 4 \left.\frac{\partial^2}{\partial \alpha^2}\right|_{\alpha=1} \int dx \, e^{-\alpha x^2/2}$$

$$= 4 \left.\frac{\partial^2}{\partial \alpha^2}\right|_{\alpha=1} \left(\text{[redacted]} \, 1/\sqrt{\alpha} \right)$$

$$= \cancel{4} \, \frac{3}{\cancel{4}}$$ (π via normalization)

→ "3" appears from moments of Gaussian

→ Moments replace/underly equation of state.

$$\epsilon_r(k,\omega) = 1 - \frac{\omega_p^2}{\omega^2}\left(1 + 3\frac{k^2 V_{Th}^2}{\omega^2}\right)$$

so

$$\epsilon = \epsilon_R + i\,\epsilon_{IM}$$

$$\epsilon_R = 1 - \frac{\omega_p^2}{\omega^2}\left(1 + 3\frac{k^2 V_{Th}^2}{\omega^2}\right)$$

$$\epsilon_{IM} = -\pi \frac{\omega_p^2}{k|k|} \frac{\partial \langle f \rangle}{\partial v}\Big|_{\omega/k}$$

$\rightarrow \epsilon_R = 0 \quad \Rightarrow$ Collective Resonance / Wave

- as ϵ derived via $(kv/\omega) < 1$ expansion, need determine $\omega(k)$ iteratively

$$\epsilon_r = 0 = 1 - \frac{\omega_p^2}{\omega^2}\left(1 + 3\frac{k^2 V_{Th}^2}{\omega^2}\right)$$

Lowest order: $\omega^{(0)} = \omega_p$

$$\Rightarrow \quad \epsilon_r = 1 - \frac{\omega_p^2}{\omega^2}\left(1 + 3\frac{k^2 V_{Th}^2}{\omega_p^2}\right)$$

$\therefore \quad \omega^2 = \omega_p^2(1 + 3k^2\lambda_D^2) \longrightarrow$ structure agrees with fluid mdl.

$\hookrightarrow$ contrast fluid

- Distribution function determines equation of state

i.e. # 3 $\leftrightarrow \int v^4 \langle f \rangle$

Contrast K + T : $\left\{ \begin{array}{l} p = p_0 (\rho/\rho_0)^{\gamma} \qquad \gamma = 3 \\ \gamma = 3 \leftrightarrow \text{Maxwellian} \end{array} \right.$

- Structure of dispersion relation identical to warm fluid model
$\Leftrightarrow k v_{Th} < \omega$

$\rightarrow \epsilon_{IM}$.

$$\epsilon_{IM} = - \pi \frac{\omega_p^2}{k|k|} \frac{\partial \langle f \rangle}{\partial v}\bigg|_{\omega/k}$$

$$Q = \omega\, \epsilon_{IM} \left(|E|^2 / 8\pi \right) \quad \rightarrow \text{dissipated energy}$$

$\Rightarrow$

$$Q = - \omega_k \frac{\pi \omega_p^2}{k|k|} \frac{\partial \langle f \rangle}{\partial v}\bigg|_{\omega_k/k} |E|^2/8\pi$$

Now, $$\frac{\partial W_k}{\partial t} + \nabla \cdot S_k + Q_k = 0$$

$$\Rightarrow \gamma_k = -Q_k / W_k \qquad W_k = \omega_k \frac{\partial \epsilon_r}{\partial \omega}\bigg|_{\omega_k} \frac{|E|^2}{8\pi}$$

$$\therefore \quad \gamma_k = \left(\frac{\pi \omega_p^2}{k|k|} \frac{\partial \langle f \rangle}{\partial v}\bigg|_{\frac{\omega_k}{k}} \right) \bigg/ \left(\frac{\partial \epsilon_r}{\partial \omega}\bigg|_{\omega_k} \right)$$

Alternatively:

$$\epsilon = \epsilon_R(\underline{k}, \omega) + i\epsilon_{IM}(\underline{k}, \omega)$$

$$\omega = \omega_{\underline{k}} + i\gamma_{\underline{k}} \qquad \gamma \ll \omega_k$$

$$\epsilon = \epsilon_R(k, \omega_k + i\gamma_k) + i\epsilon_{IM}(k, \omega_k)$$

$$\cong \epsilon_R(k, \omega_k) + i\gamma_k \frac{\partial \epsilon_R}{\partial \omega}\bigg|_{\omega_k} + i\epsilon_{IM}(k, \omega_k)$$

$$\gamma_{\underline{k}} = -\epsilon_{IM}(k, \omega_k) \big/ (\partial \epsilon_R / \partial \omega)\big|_{\omega_k}$$

agrees above.

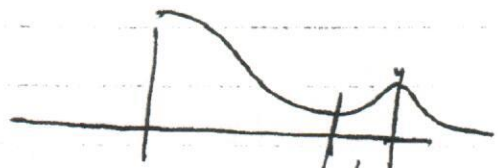
Thus $\rightarrow$ $\partial\langle f\rangle/\partial v\big|_{\omega_k/k} < 0$

$\Rightarrow$ damping (Landau damping)

$\rightarrow$ $\partial\langle f\rangle/\partial v\big|_{\omega/k} > 0$

$\Rightarrow$ growth

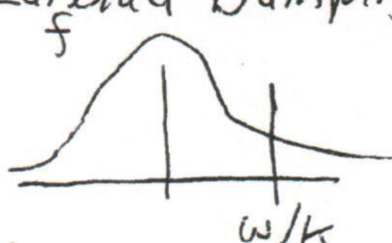
i.e. 'Bump on Tail'



$\omega/k \sim v$ grows
as $\partial\langle f\rangle/\partial v > 0$

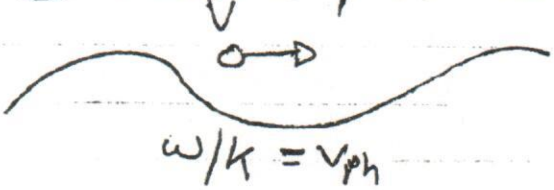
$\rightarrow$ Physics of Landau Damping

Consider



$\rightarrow$ Landau damping occurs due
wave particle resonance $\omega/k \sim v$

$\rightarrow$ intuitively, consider wave interaction
with ~ resonant particle



Resonant particle 'sees' ~ DC field

$$\frac{dv}{dt} = \frac{q}{m} E \cos(kx-\omega t)$$

$$= \frac{q}{m} E \cos\left(k\left(x - v_{ph} t\right)\right)$$

if boost to frame at particle velocity V

$x' = x - Vt$
$v' = v - V$
$a' = a$

$\Rightarrow$

$$\frac{dv}{dt} = \frac{q}{m} E \cos\left(k\left(x + (V - v_{ph})t\right)\right)$$

$\therefore$ – secular (in time) interaction at $V \sim v_{ph}$ resonance

– $v \lesssim \omega/k \Rightarrow$ wave accelerates particles, loses energy

$v \gtrsim \omega/k \Rightarrow$ waves decelerates particles, gains energy

Q = # accelerated – # decelerated

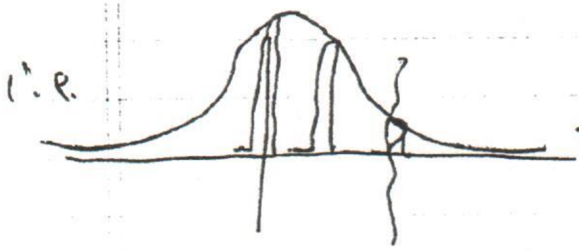
$\sim (\partial f/\partial v)\big|_{\omega/k}$

▷ Quantitatively :

— as $Q = \langle \underline{E}^* \cdot \underline{J} \rangle$

seek $\bar{q} = \langle q v \underline{E} \rangle$ → time averaged work on resonant 'beam'

i.e.



⇒ plasma distribution as superposition of beams

then $Q = \int dv\, \bar{q}$

— $v = v_0 + \delta v$ ↦ perturbations induced by wave

$x = x_0 + \delta x$

so
$$\frac{d\delta v}{dt} = \frac{q}{m} E\Big|_{x_0, v_0}$$

$$\frac{d\delta x}{dt} = \delta v$$

$$\bar{q} = q\langle vE \rangle$$

$$v = v_0 + \delta v$$
$$E = E(t, x = x_0 + \delta x)$$
$$\cong E(t, x_0) + \delta x\, \partial E / \partial x\big|_{x_0}$$

$$\bar{q} = q\left\langle \left(\overset{\text{DC}}{V_0} + \overset{\text{osc}}{\delta V}\right)\left(\overset{\text{osc}}{E(t,x_0)} + \overset{\text{both osc.}}{\delta x\, \partial E/\partial x\Big|_{x_0,t}}\right)\right\rangle$$

$$\bar{q} = q V_0 \left\langle \delta x\, \partial E/\partial x\Big|_{x_0,t} \right\rangle + q\left\langle \delta V\, E(t,x_0)\right\rangle$$

Now, $\dfrac{d\delta V}{dt} = \dfrac{q}{m} E(t,x_0)$ $\qquad x_0 = x_0' + V_0 t$

$$= \frac{q}{m} E_0\, e^{ikx_0'}\, e^{ik(V_0 - \omega/k)t}\, e^{\delta t}$$

$x_0' = 0$ (convenience)

$\omega/k = v_{ph}$

$\delta > 0 \Rightarrow \delta V \to 0$ as $t \to -\infty$

$\therefore$

$$\frac{d\delta V}{dt} = \frac{q}{m} E_0 \exp\left(ik\left(V_0 - \omega/k - i\delta\right)t\right)$$

$$\delta V = \frac{q}{m} \frac{E_0\, e^{ik(V_0 - \omega/k - i\delta)t}}{i\left(k(V_0 - v_{ph}) - i\delta\right)}\Bigg|_{-\infty}^{t}$$

$$\Rightarrow \delta V = \frac{q}{m} E(t,x_0) \Big/ ik\left(V_0 - v_{ph}\right) + \delta$$

$$\delta x = \frac{q}{m} E(t,x_0) \Big/ \left(ik\left(V_0 - v_{ph}\right) + \delta\right)^2$$

Thus

$$\bar{q} = q v_0 \left\langle \delta x \frac{\partial E^*}{\partial x} \right\rangle + q \left\langle \delta v\, E \right\rangle$$

$$= q v_0 \left\langle -ik\, E^*(t, x_0)\, \frac{q}{m} \frac{E(t, x_0)}{(ik(v_0 - v_p) + \delta)^2} \right\rangle$$

$$+ q \left\langle E^*(t, x_0)\, \frac{q}{m} \frac{E(t, x_0)}{(ik(v_0 - v_p) + \delta)} \right\rangle$$

note: $E^* E$ gives DC beat

$\Rightarrow$

$$\bar{q} = \frac{d}{dv_0} \left\{ \frac{q^2}{2\cancel{k}m} |E|^2 \frac{v_0}{[ik(v_0 - v_p) + \delta]} \right\}$$

$$= \frac{d}{dv_0} \left\{ \frac{q^2}{2\cancel{k}m} |E|^2 \frac{-i v_0}{k(v_0 - v_p) - i\delta} \right\}$$

real part $\Rightarrow$

$$\bar{q} = \frac{d}{dv_0} \left\{ \frac{q^2}{2\cancel{k}m} |E|^2 \frac{v_0 \pi}{|k|} \delta(v_0 - v_p) \right\}$$

$$Q = n \int dv_0 \, \bar{q}(v_0) \langle f(v_0) \rangle$$

$$= \int dv_0 \langle f(v_0) \rangle \frac{d}{dv_0} \left\{ \frac{nq^2 |E|^2}{2m} \frac{v_0 \pi}{|k|} \delta(v_0 - v_{ph}) \right\}$$

$$= -\frac{\pi \omega_p^2}{|k|} \frac{\omega}{k} \frac{\partial \langle f(v) \rangle}{\partial v} \bigg|_{\omega/k} \left(|E|^2 / 8\pi \right)$$

$\frac{1}{2} \rightarrow \frac{4\pi \ldots}{8\pi \ldots}$

$\Rightarrow$

$$\boxed{Q = -\pi \frac{\omega_p^2}{|k|} \frac{\omega}{k} \frac{\partial \langle f \rangle}{\partial v} \bigg|_{\omega/k} \left(|E|^2 / 8\pi \right)}$$

$\rightarrow$ agrees with previous result

$\rightarrow$ establishes Landau damping mechanism as collisionless heating, due secularity at wave-particle resonance. Corresponds to E·J work of electric field on particles.

$\rightarrow$ Fate of energy :

$$\frac{\partial W_n}{\partial t} + \cancel{\nabla \cdot S_n} + Q_n = 0$$

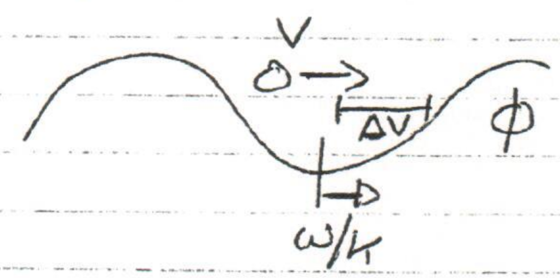
$$\frac{\partial W_n}{\partial t} = -Q_n \qquad \Rightarrow \text{L.D.} \leftrightarrow \text{wave energy dissipated}$$

but clearly resonant particles heated

so $\frac{\partial \, RPKED}{\partial t} + \frac{\partial}{\partial t} W_k = 0$

$\therefore$ $\boxed{\text{Landau damping heats resonant piece of distribution at expense of wave energy}}$.

$\Rightarrow$ Clearly, linear theory of Landau damping only valid for times less than bounce time in trough of wave:



$\Delta V \sim \sqrt{q\phi/m}$

$1/\tau_b = k \Delta V$

Then $\gamma_k = \gamma_k^{(0)}$ for $t < \tau_b$, only.

$\Rightarrow$ Landau damping forces/drives a picture of plasma gas of:

- resonant particles and waves
- waves — collective modes consisting of non-resonant particles + fields